A LAMAN THEOREM FOR FRAMEWORKS ON SURFACES OF REVOLUTION

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Abstract. A foundational theorem of Laman provides a counting characterisation of the finite simple graphs whose generic bar-joint frameworks in two dimensions are infinitesimally rigid. Recently a Laman-type characterisation was obtained for frameworks in three dimensions whose vertices are constrained to concentric spheres or to concentric cylinders. Noting that the plane and the sphere have 3 independent locally tangential infinitesimal motions while the cylinder has 2, we obtain here a Laman-Henneberg theorem for frameworks on algebraic surfaces with a 1-dimensional space of tangential motions. Such surfaces include the torus, helicoids and surfaces of revolution. The relevant class of graphs are the $(2,1)$-tight graphs, in contrast to $(2,3)$-tightness for the plane/sphere and $(2,2)$-tightness for the cylinder. The proof uses a new characterisation of simple $(2,1)$-tight graphs and an inductive construction requiring generic rigidity preservation for 5 graph moves, including the two Henneberg moves, an edge joining move and various vertex surgery moves.

1. Introduction

A bar-joint framework in real Euclidean space $\mathbb{R}^d$ is a geometric realisation of the vertices of a graph with the edges considered as inextensible bars between them. Such a framework is said to be rigid if there is no non-trivial continuous motion of the framework vertices which maintains bar-lengths, and is said to be flexible if it is not rigid. A foundational theorem of Laman, obtained in 1970, asserts that the rigidity of a generically positioned framework in the plane depends only on the underlying graph and furthermore these graphs are characterised in terms of a simple counting condition. There is also an elegant recursive construction of the minimally rigid frameworks going back to Henneberg [6], [9] in which each framework may be derived from a single edge framework by repeated application of two simple construction moves, namely the Henneberg 1 move and the Henneberg 2 move. Analogous characterisations for frameworks in $\mathbb{R}^3$ remain open problems and no combinatorial characterisation of infinitesimal rigidity is known. We note however that a number of partial and related results are given in Whiteley [23] and that the longstanding molecular conjecture has been resolved by Katoh and Tanigawa [8].

Attention has also been given to frameworks in 3-dimensional space whose vertices are constrained to 2-dimensional surfaces. In [13] we obtained Laman-Henneberg-type theorems, which we also reprove below, for the case of constraint to parallel planes, concentric spheres and concentric cylinders. The cylinder case presents new complications both for

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the purely graph theoretical analysis and for the preservation of rigidity under the Henneberg moves and the further construction moves that are needed.

A plane surface in $\mathbb{R}^3$ has a three-dimensional vector space of internal infinitesimal motions, coming from translations and rotations, while the (infinite circular) cylinder has two such independent motions. More formally, in Definition 2.2 we define the type k of an irreducible surface where k takes values $3, 2, 1$ or 0. (See also Definition 7.3.) The type of a surface is reflected in the graph counting conditions for Laman type theorems; fewer independent infinitesimal motions for the surface imply a richer set of graphs which in turn require more constructive moves and more refined proof techniques. The main result in the present paper is the following Laman-Henneberg-type theorem for bar-joint frameworks in $\mathbb{R}^3$ whose vertices are constrained to an algebraic surface $\mathcal{M}$ of type 1. These surfaces include a range of fundamental algebraic surfaces such as the elliptical cylinder, the cone, the torus, surfaces of revolution, and various helical glide-translation surfaces.

Theorem 1.1. *Let $G = (V, E)$ be a simple graph, let $\mathcal{M}$ be an irreducible algebraic surface in $\mathbb{R}^3$ of type 1 and let (G, p) be generic framework on $\mathcal{M}$. Then (G, p) is isostatic on $\mathcal{M}$ if and only G is K_1, K_2, K_3, K_4 or is $(2, 1)$-tight.*

For a plane surface the required graphs are the simple graphs known as Laman graphs, corresponding to the top count $|E| = 2|V| - 3$ and the inequality $|E'| \leq 2|V'| - 3$ for every subgraph (V', E'). In fact these are the $(2, 3)$-tight simple graphs. For the cylinder the appropriate graphs are the $(2, 2)$-tight graphs and once again the graphs are necessarily simple. The $(2, 1)$ tight simple graphs were characterised recently in Nixon and Owen [14] and we describe this and associated characterisations and construction moves in Sections 2 and 3. Moreover, we obtain the following new characterisation of $(2, 1)$-tight simple graphs which turns out to be efficient for our purposes. The methods for this led to a new analogous characterisation of simple $(2, 2)$-tight graphs given in Section 3.

Theorem 1.2. *A simple finite graph is $(2, 1)$-tight if and only if it is equal to $K_5\backslash e$ or can be obtained from this graph by the sequential application of moves of 5 types, namely the Henneberg 1 and 2 moves, the vertex-to-K_4 move, the vertex-to-4-cycle move, and the edge joining move.*

The proof of the main theorem is principally concerned with the sufficiency for rigidity of the combinatorial condition and we obtain this by showing that each of the moves in the constructive sequence for the graph preserves infinitesimal rigidity. We introduce some new methods for this and there are two moves that present particular challenges, namely the Henneberg 2 move and the vertex-to-K_4 move. For the Henneberg 2 move we give two quite different proofs which also give new proofs in the case of the circular cylinder. The first of these uses a convergence argument involving a sequence of generic realisations of G' which converge to a degenerate nongeneric realisation of G' which in a natural sense covers a generic realisation of G. On the other hand in the Section 5 we adopt the traditional approach of algebraic specialisation to obtain a direct entirely algebraic proof.

For the vertex-to-K_4 move we show that proper flexes are inherited under the inverse move $G' \to G$ corresponding to K_4 contraction to a vertex. This is achieved through the consideration of a sequence (G', p^k) in which the K_4-subframeworks contract in a manner which is *well-behaved* with respect to the distinct principal curvatures of the surface.

In all our considerations the framework vertices are constrained to a surface while the edges are straight Euclidean edges measured by distances in $\mathbb{R}^3$. We note that Whiteley [21] has considered the topic of frameworks on surfaces with geodesic edges and there the appropriate combinatorial objects are looped multigraphs. Moreover, rigidity is expressed in terms of the rank of the k-frame matrix rather than the rigidity matrix. We note also that frameworks on surfaces with geodesic bars are also implicit in the context of periodic frameworks, considered, for example, by Borcea and Streinu [3], Malestein and Theran [10], [11], Nixon and Ross [16], Owen and Power [17] and Ross [18].

The paper is structured as follows. In Section 2 we recall basic definitions and key results from [13] for generic frameworks on surfaces. In Section 3 we detail the inductive moves on graphs and obtain inductive characterisations of simple (j,k)-tight graphs. In Section 4 we prove the preservation of generic independence for the two Henneberg moves as moves on frameworks on algebraic surfaces of type 1. In Section 5 we prove the preservation of generic independence for various vertex surgery moves, including vertex splitting, vertex-to-4-cycle and vertex-to-K_4. In Section 6 we provide an alternative proof of generic independence under Henneberg 2 moves that we believe could be of independent interest. In the final section we prove the main theorem and indicate some further results and directions.

2. Frameworks on Surfaces

Let $\mathcal{M}$ be a subset of $\mathbb{R}^3$ which is a smooth surface in the sense of being a 2-dimensional embedded differentiable manifold. The main examples we have in mind are defined as disjoint unions of parts of elementary algebraic surfaces. Accordingly we assume smoothness in the sense that local coordinate maps exist for $\mathcal{M}$ which are analytic. In particular for every point of $\mathcal{M}$ there is a continuous choice of normal vectors in a neighbourhood of the point and a Taylor series expansion, as in Equation 5.1, for points of $\mathcal{M}$ in this neighbourhood.

A framework (G,p) on $\mathcal{M}$ is a finite bar-joint framework in $\mathbb{R}^3$, for a simple graph $G=(V,E)$, with framework points $p(v), v \in V$, which lie on $\mathcal{M}$.

An *infinitesimal flex* of (G,p) on $\mathcal{M}$ is a sequence or vector u of velocity vectors $u_1,\dots,u_{|V|}$, considered as acting at the framework points, which are tangential to the surface and satisfy the infinitesimal flex requirement in $\mathbb{R}^3$, namely

$$u_i.(p_i-p_j)=u_j.(p_i-p_j),$$

for each edge v_iv_j. It is elementary to show that u is an infinitesimal flex if and only if u lies in the nullspace (kernel) of the rigidity matrix $R_{\mathcal{M}}(G,p)$ given in the following definition. The submatrix of $R_{\mathcal{M}}(G,p)$ given by the first $|E|$ rows provides the usual rigidity matrix, $R_3(G,p)$ say, for the unrestricted framework (G,p). The tangentiality condition corresponds to u lying in the nullspace of the matrix formed by the last $|V|$ rows.

Definition 2.1. *The* rigidity matrix $R_{\mathcal{M}}(G,p)$ *of* (G,p) *on* $\mathcal{M}$ *is an* $|E|+|V|$ *by* $3|V|$ *matrix in which consecutive triples of columns correspond to framework points. The first* $|E|$ *rows correspond to the edges and the entries in row* $e=uv$ *are zero except possibly in the column triples for* $p(u)$ *and* $p(v)$*, where the entries are the coordinates of* $p(u)-p(v)$ *and* $p(v)-p(u)$ *respectively. The final* $|V|$ *rows correspond to the vertices and the entries*

in the row for vertex v are zero except in the columns for v where the entries are the coordinates of a normal vector $N(p(v))$ to $\mathcal{M}$ at $p(v)$.

The case of a surface $\mathcal{M}$ which is a subset of the nonsingular points of a polynomial equation $m(x, y, z) = 0$ is of particular interest, especially when $m(x, y, z)$ is irreducible over some coefficient field. In the sequel we confine attention to the rational field and refer to such surfaces simply as *irreducible surfaces*. In this case we may take the derivative of $m(x, y, z)$ at $p(v)$ for the choice of normal $N(p(v))$. Furthermore, the rigidity matrix arises from the derivative of the augmented edge-function $\tilde{f}_G$, with

$$2R_{\mathcal{M}}(G, p) = (D\tilde{f}_G)(p),$$

where $\tilde{f}_G : \mathbb{R}^{3|V|} \to \mathbb{R}^{|E|+|V|}$ is given by $\tilde{f}_G(q) = (f_G(q), m(q_1), \ldots, m(q_{|V|}))$ with

$$f_G(q) = (\|q_i - q_j\|^2)_{v_iv_j \in E}$$

the usual edge function for G associated with framework realisations in $\mathbb{R}^{3n}$, where $\|.\|$ is the usual Euclidean norm.

As is well-known, for $n \geq 4$ a complete graph framework (K_n, p) in $\mathbb{R}^3$, not lying in a hyperplane, has a 6-dimensional vector space of infinitesimal flexes, a basis for which may be provided by a set of linearly independent infinitesimal flexes associated with translations and rotations. When the vertices of (K_n, p) are constrained to a surface $\mathcal{M}$ then the dimension is reduced to $\dim \ker R_{\mathcal{M}}(K_n, p) = k$ where $k = 3, 2, 1$ or 0.

We now define smooth surfaces of type k for $k = 3, 2, 1, 0$. The type number reflects the number of independent infinitesimal motions of a typical framework on $\mathcal{M}$ that arise from isometries of $\mathbb{R}^3$ that act tangentially at every point on $\mathcal{M}$ (not just the framework joints). For the sphere, cylinder and cone the types are $3, 2$ and 1 respectively, while the ellipsoid, defined by $x^2 + 2y^2 + 3z^2 = 1$, has type 0.

Definition 2.2. *A surface $\mathcal{M}$ is said to be of type k, or to have freedom number k, if* $\dim \ker R_{\mathcal{M}}(K_n, p) \geq k$ *for all complete graph frameworks (K_n, p) on $\mathcal{M}$ and k is the largest such number.*

Apart from the type 3 surfaces, which arise from concentric spheres or parallel planes, a typical K_4 framework on a surface $\mathcal{M}$ has a two-dimensional space of infinitesimal flexes. This follows on consideration of the 10 by 12 rigidity matrix. For K_3 and K_2 frameworks the space is three-dimensional and includes rotational flexes not derivable from (tangentially acting) isometries. For the cylinder the flexes of K_4 frameworks are all associated with isometries whereas on the cone (resp. ellipsoid) there is a one-dimensional (resp. zero-dimensional) subspace determined by tangential isometries.

Definition 2.3. *Let $\mathcal{M}$ be a smooth surface and $p = (p_1, \ldots, p_n)$ a vector of points on $\mathcal{M}$. Then the framework (G, p) on $\mathcal{M}$ is said to be infinitesimally rigid if every infinitesimal flex of (G, p) on $\mathcal{M}$ corresponds to a rigid motion flex of $\mathbb{R}^3$. In particular if* $\dim \ker R_{\mathcal{M}}(K_n, p)$ *agrees with the freedom number k of $\mathcal{M}$ then (G, p) is infinitesimally rigid on $\mathcal{M}$ if and only if*

$$\dim \ker R_{\mathcal{M}}(G, p) = k.$$

A framework (G, p) on $\mathcal{M}$ is *independent* if $R_{\mathcal{M}}(G, p)$ has linearly independent rows and is *minimally infinitesimally rigid on* $\mathcal{M}$, or *isostatic on* $\mathcal{M}$ if it is independent and infinitesimally rigid on $\mathcal{M}$.

From the point of view of the infinitesimal rigidity it is only the nature of $\mathcal{M}^{|V|}$ in a neighbourhood of p which is of significance. On the other hand for irreducible surfaces one can establish generic properties for the pair $G, \mathcal{M}$ as we shall see.

Following Asimow and Roth we say that a framework (G, p) on $\mathcal{M}$ is *regular* if the rank of $R_{\mathcal{M}}(G, q)$ takes its maximum value throughout a neighbourhood of p in $\mathcal{M}^{|V|}$. In the case that $\mathcal{M}$ is an algebraic surface determined by an irreducible polynomial $m(x, y, z)$ over $\mathbb{Q}$, the framework (G, p) is said to be *generic* on $\mathcal{M}$ if the field extension $\mathbb{Q}(p) : \mathbb{Q}$ has transcendence degree $2|V|$. This is to say that an algebraic dependency $h(\{x_i\}, \{y_i\}, \{z_i\}) = 0$ holds between the coordinates x_i, y_i, z_i of all the points p_i only when the polynomial $h(\{X_i\}, \{Y_i\}, \{Z_i\})$ lies in the ideal generated by the polynomials $m(X_i, Y_i, Z_i), 1 \leq i \leq |V|$. It is a standard exercise to show that such generic frameworks are regular.

Note that in contrast to type 3 and 2 there are diverse classical surfaces of type 1, including spheroids, with isometry group S^1, elliptical cylinders and other noncircular cylinders, with translational isometry group $\mathbb{R}^1$, and circular hyperboloids and other diverse surfaces with glide-rotation isometry group $\mathbb{R}^1$.

Definition 2.4. *A simple graph G is independent for the irreducible surface $\mathcal{M}$ if every generic framework (G, p) on $\mathcal{M}$ is independent.*

In particular, K_4 is dependent for the sphere but independent for the cylinder. On the other hand $K_5 \backslash e$ is dependent for the cylinder but independent for the cone. Note that $K_n \backslash e$ denotes the unique graph formed by deleting any single edge from K_n.

The determination of combinatorial conditions for the generic independence of classes of frameworks is one of the fundamental problems in constraint system rigidity theory. See for example Whiteley [22], [23] and Jackson and Jordan [7]. Our main result can be viewed in this spirit. Also we note that there is the following matroidal interpretation of our main result. Let $\mathcal{R}(K_n, \mathcal{M})$ be the linear matroid for the rigidity matrix $R_{\mathcal{M}}(K_n, p)$ associated with a generic n-tuple and the irreducible surface $\mathcal{M}$. Then by Theorem 1.1 the bases of $R_{\mathcal{M}}(K_n, p)$ correspond to sets of rows determined by the $(2, 1)$-tight subgraphs.

The notions of continuous rigidity and minimal continuous rigidity are also naturally defined in the surface setting and the following equivalence from [13] is an analogue of a theorem of Gluck [5].

Theorem 2.5. *A generic framework (G, p) on an algebraic surface $\mathcal{M}$ is infinitesimally rigid if and only if it is continuously rigid on $\mathcal{M}$.*

In the sequel the infinitesimal rigidity perspective will be more direct and we make use of the following two results from [13], namely a version of the Maxwell counting condition and an isostatic characterisation in the spirit of Asimow and Roth [1].

Theorem 2.6. *Let (G, p) be an isostatic generic framework on the algebraic surface $\mathcal{M}$ of type $k, 0 \leq k \leq 3$, with G not equal to K_1, K_2, K_3 or K_4. Then $|E| = 2|V| - k$ and for every subgraph H of G with at least one edge, $|E(H)| \leq 2|V(H)| - k$.*

Theorem 2.7. *Let (G, p) be a generic framework on a surface $\mathcal{M}$ of type k. Then (G, p) is isostatic on $\mathcal{M}$ if and only*

(1) $\operatorname{rank} R_{\mathcal{M}}(G, p) = 3|V| - k$ *and*

(2) $2|V| - |E| = k$.

The classes of graphs in Theorem 2.6 are the simple graphs that are $(2, k)$-tight, with $k = 0, 1, 2, 3$, in the following sense.

Definition 2.8. *A graph $G = (V, E)$ is (j, k)-*sparse *if for all subgraphs H the inequalities $|E(H)| \leq j|V(H)| - k$ hold. Moreover G is (j, k)-*tight *if G is (j, k)-sparse and $|E| = j|V| - k$.*

The inductive characterisations of these classes of simple graphs for $j = 2$ and $k = 3, 2$ and 1 form a key part of our approach to sufficiency of the necessary counting conditions for generic infinitesimal rigidity. We describe the various construction moves in the next section.

Theorem 2.9 (Henneberg [6] and Laman [9])**.** *A simple graph G is $(2, 3)$-tight if and only if it can be generated from K_2 by Henneberg 1 and 2 moves.*

This characterisation plays a role in the following extension of Laman's theorem.

Theorem 2.10 ([13])**.** *Let $G = (V, E)$, let $\mathcal{M}$ be a union of parallel planes or a union of concentric spheres and let p be generic on $\mathcal{M}$. Then (G, p) is isostatic on $\mathcal{M}$ if and only if G is K_1, K_2 or $(2, 3)$-tight.*

Theorem 2.11 ([13] and [14])**.** *For a simple graph G the following are equivalent:*

(1) *G is $(2, 2)$-tight,*
(2) *G can be generated from K_1 by Henneberg 1, Henneberg 2 and graph extension moves,*
(3) *G can be generated from K_1 by Henneberg 1, Henneberg 2, vertex-to-K_4 and vertex splitting moves.*

Theorem 2.12 ([13])**.** *Let $G = (V, E)$, let $\mathcal{M}$ be a cylinder or a union of concentric cylinders and let p be generic on $\mathcal{M}$. Then (G, p) is isostatic on $\mathcal{M}$ if and only if G is K_1, K_2, K_3 or is $(2, 2)$-tight.*

We observe that this theorem can also be proven, using the methods of this paper, by applying the equivalence of (1) and (3) in Theorem 2.11 or by applying Theorem 3.2.

Let $K_n \sqcup K_m$ denote the unique graph which is the join of K_n and K_m over a single common edge and its two vertices. The following theorem provides an alternative to Theorem 1.2 with vertex splitting replacing the vertex-to-4-cycle move.

Theorem 2.13 ([14])**.** *A simple graph G is $(2, 1)$-tight if and only if it can be generated from $K_5 \backslash e$ or $K_4 \sqcup K_4$ by Henneberg 1, Henneberg 2, vertex-to-K_4, vertex splitting and edge joining moves.*

3. Simple $(2, 1)$-tight graphs

In this section we prove Theorem 1.2 and in Theorem 3.2 we obtain an analogous characterisation of $(2, 2)$-tight simple graphs. We remark that the insistence on simplicity makes the construction problematic. Indeed if we permit loops and parallel edges then a general constructive theorem of Fekete and Szego [4] applies. In the case of $(2, 1)$-tight graphs it ensures that the only operations required are Henneberg 1 and 2 type

operations. Fekete and Szego's result extended work of Tay on (k, k)-tight graphs for the characterisation of the generic rigidity of body-bar frameworks in arbitrary dimension [19].

The inductive characterisation of $(2, 3)$-tight graphs of Henneberg and Laman starts with the elementary counting observation that for $2 \leq k \leq 3$ the average degree in a $(2, k)$-tight graph is less than 4, while there can be no vertices of degree less than 2. Thus degree 2 or degree 3 vertices exist. Henneberg introduced the following two operations on graphs which maintain the sparsity count, increase the vertex count by 1, and add a new vertex of degree 2 or degree 3 respectively. See also the discussions in [9], [15] and [20]

The *Henneberg* 1 move augments a graph G by adding a a vertex v of degree 2 and two edges vv_1, vv_2 from it to distinct neighbours v_1, v_2 in G,

The *Henneberg* 2 move removes an edge v_1v_2 from a graph and adds a vertex v of degree 3 with distinct neighbours v_1, v_2, v_3 for some vertex v_3. This is also refered to as edge splitting.

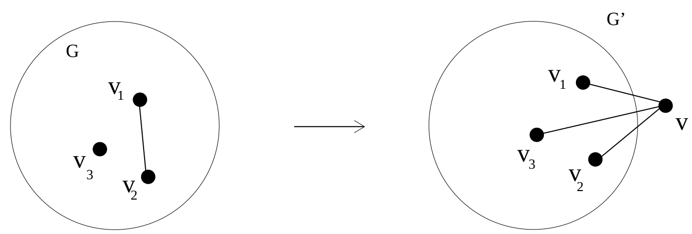

FIGURE 1. The Henneberg 2 move.

Laman proved that every $(2, 3)$-tight graph can be generated recursively from K_2 by a sequence of these operations. The key step is to show that an inverse Henneberg 2 operation is possible on a degree 3 vertex v, by virtue of the fact that at least one of the three choices for the new edge (v_1v_2, v_2v_3 or v_3v_1) can be made without violating any subgraph count.

On the other hand, for a $(2, 2)$-tight graph it is easy to see that degree 3 vertices may be contained within subgraphs isomorphic to K_4 and so admit no inverse Henneberg 2 move. Indeed there are countably many $(2, 2)$-tight graphs for which every vertex of degree less than 4 is contained in a copy of K_4. In view of this obstruction, in [13] and [14] we considered additional graph moves preserving $(2, 2)$-tightness, including those indicated in Figure 2 and Figure 3. These are the *vertex-to-K_4 move* and the *vertex splitting move.*

The vertex-to-K_4 move substitutes a copy of K_4 in place of a vertex v, with an arbitrary replacement of edges xv by edges xw with w in $V(K_4)$. More generally, the *graph extension move* performs similar surgery with v replaced by a $(2, 2)$-tight graph. The inverse move associated with the vertex-to-K_4 move we refer to as an allowable K_4-to-vertex move, or simply as an allowable K_4 contraction.

The *vertex splitting* operation removes an edge uv and vertex v and inserts a copy of K_3 on vertices u, v_1, v_2 and assigns all edges xv into either an edge xv_1 or an edge xv_2. That

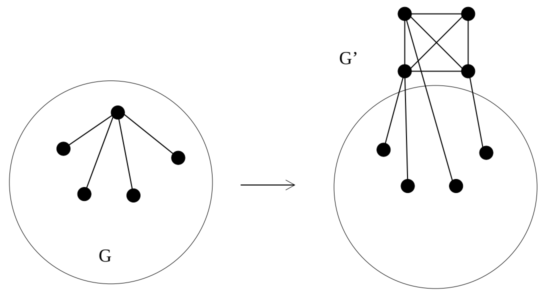

FIGURE 2. The vertex-to-K_4 move.

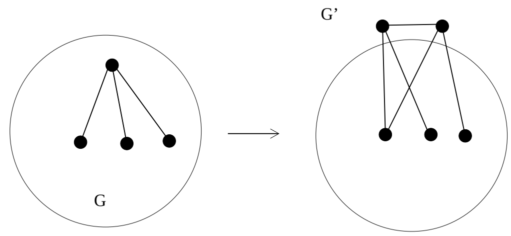

FIGURE 3. The vertex splitting move.

is, let $G = (V, E)$, let $V^* := V - v$, let $E^* := E(G[V^*])$ and let $E' := \{xy \in E : y = v\}$. Thus $G = (V, E) = (V^* + v, E^* + E' + uv)$ and if G' is the result of a vertex split on the edge uv then $G' = (V', E') = (V^* + \{v_1, v_2\}, E^* + E'_1 + E'_2 + \{uv_1, uv_2, v_1v_2\})$ where $E'_i := \{xy \in E' : x = v_i\}$ is an arbitrary partition of E'. We refer to the associated inverse move as an allowable edge contraction or an admissible triangle contraction.

Finally, an edge-joining move [14] combines two graphs G, H to form a new graph with the vertices and edges of these graphs together with an additional connecting edge $e = gh$ with g in G and h in H.

The proof of the inductive characterisation of $(2,1)$-tight graphs given in Theorem 2.13 is proven along the following lines. By simple counting, if there are no inverse Henneberg 1 moves then there are at least two vertices of degree 3. If there are also no inverse Henneberg 2 moves then it is shown that all such vertices are in subgraphs which are copies of K_4. In this case there is an admissible K_4-contraction move unless there are triangle obstructions in the sense of there being inclusions $K_4 \to K_4 \sqcup K_3$. In this case contraction of the K_4 graph violates simplicity. However, if triangle obstructions persist this leads to the existence of an admissible triangle contraction, that is, to an inverse vertex splitting move. In this way one can arrive at the following key lemma of [14]) which in turn leads to Theorem 2.13.

Lemma 3.1. *Let G be a $(2,1)$-tight graph which contains a copy of K_3. Then either $G = K_4$, G has an allowable inverse vertex-splitting move, an allowable K_4 contraction, or every copy of K_3 is in a copy of $K_4 \sqcup K_4$ or $K_5 \backslash e$.*

We now prove Theorem 1.2 which gives a somewhat more efficient characterisation of simple $(2,1)$-tight graphs.

The vertex-to-4-cycle move is a certain vertex splitting operation, as in Figure 4. The vertex v_1 is split to two vertices v_1 and v_0 and the edges v_1v_2 and v_1v_3 are duplicated as

v_0v_2 and v_0v_3. Other edges of the form vv_1 are either left or are replaced by vv_0. The move preserves $(2, k)$-tightness and after the vertex split there will be no edge pair wv_1, wv_0 with $w \neq v_2, v_3$ and no edge v_0v_1.

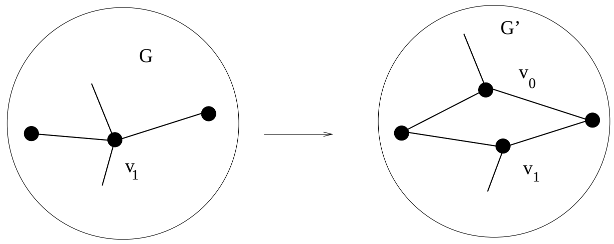

FIGURE 4. The vertex-to-4-cycle move.

The reverse of such a move on a simple $(2, 1)$-tight graph will be referred to as an admissible 4-cycle contraction.

Proof of Theorem 1.2. The usual incidence degree counting argument shows that if G is $(2, 1)$-tight with no Henneberg 1 or 2 moves then there are at least two degree 3 vertices. Indeed, if there are n_k vertices of degree k then we have $3n_3 + \cdots + 4n_4 + \cdots + rn_r = 2|E| = 2(2|V| - 1)$ and so

$$\sum_{k=1}^{r}(r-4)n_k = -2.$$

In fact these vertices must be in copies of K_4 graphs. If not then there is a potential inverse Henneberg 2 move and a counting argument (in the style of Laman [9]) shows that these inverse moves are admissible. (See also [13] and [14].)

Suppose now that one of these K_4 graphs does not have an admissible contraction to a vertex. Then there is a vertex w outside this K_4 with two edges to distinct vertices a, b in this K_4, neither of which is of degree 3. Thus there is a 4-cycle including w, a, b and a degree 3 vertex v of the K_4 graph. The edge vw is absent from G and so there is a potential inverse 4-cycle move, with $v \to w$, as long as the edge wc is absent, where c is the fourth K_4 vertex, since the move creates this edge.

Suppose first that wc is absent. We claim that the inverse 4-cycle move is admissible, that is, the result is $(2, 1)$-tight as well as being simple.

To check this we need only consider the count change for subgraphs Y of G, with $f(Y) = 2|V(Y)| - |E(Y)| = 1$, which include w and c (and so receive wc) but not v (since otherwise the count change is zero). There are three cases:

(i) a and b are in Y. However, adding v and 3 edges would create Y^+ with $f(Y^+) = 0$, so Y cannot have $f(Y) = 1$.

(ii) a and b are not in Y. However, adding a and b and their 5 edges aw, ac, bc, bw, ab to Y creates Y^+ with $f(Y^+) = 0$.

(iii) Just b (or a) is in Y. However, adding a and the 3 edges ac, aw, ab creates Y^+ with $f(Y^+) = 0$.

Thus the inverse move is admissible.

Suppose now that the edge wc is present. Then G contains a copy of $K_5\backslash e$ supported by the vertices v, w, a, b, c. We show that if G is not equal to this subgraph then there is an inverse edge-joining move. This completes the proof of the theorem.

We argue as in Lemma 4.10 of [14]. Let $Y = \{Y_1, \dots, Y_n\}$ be the subgraphs which are copies of $K_5\backslash e$. They are necessarily vertex disjoint since $f(Y_i \cup Y_j) = 2 - f(Y_i \cap Y_j)$ and every proper subgraph X of $K_5\backslash e$ has $f(X) \geq 2$. Let V_0 and E_0 be the sets of edges in G which are in none of the Y_i. Then

$$f(G) = \sum_{i=1}^{n} f(Y_i) + 2|V_0| - |E_0|$$

so $E_0 = 2|V_0| + n - 1$. Each vertex in V_0 is incident to at least 4 edges. If every Y_i is incident to at least 2 edges in E_0 then there are at least $4|V_0| + 2n$ edge/vertex incidences in E_0. This implies $|E_0| \geq 2|V_0|+n$, a contradiction. Thus either there is a copy Y_i with no incidences, which would imply $G = Y_i$, since G is connected, contrary to our assumption, or there is a copy with one incidence. In this case there is an inverse edge-join move, as desired. □

Theorem 3.2. *A simple finite graph G is $(2, 2)$-tight if and only if it can be obtained from K_1 by the sequential application of moves of 4 types, namely the Henneberg 1 and 2 moves, the vertex-to-K_4 move and the vertex-to-4-cycle move.*

Proof. Suppose that G is simple and $(2, 2)$-tight with at least one edge. Then, by simple counting, there exist vertices of degree 2 or 3. As in the last proof, if there are no inverse Henneberg moves then it follows that the minimum degree is 3 and any such vertex v must lie in a copy of K_4. As above if there is no admissible contraction on this K_4 then there is a vertex w adjacent to two vertices a, b in that copy of K_4. Since G is $(2, 2)$-sparse there is no edge wc, with c the final K_4 vertex, and hence, the argument above, shows there is an admissible 4-cycle contraction move. □

4. Henneberg moves on frameworks on surfaces

The role played by the Henneberg moves in rigidity theory is extensive and well studied (see for example [2], [6], [7], [9], [19], [23]) and we now consider such moves on frameworks on surfaces. The case of the Henneberg 1 move is elementary. Recall that an isostatic framework is one which is independent and infinitesimally rigid.

Lemma 4.1. *Let (G, p) be a generic framework on an algebraic surface $\mathcal{M}$, let G' be a graph obtained from G by a Henneberg 1 move with new vertex v, and let $p' = (p, p_v)$ be generic. Then (G, p) is isostatic on $\mathcal{M}$ if and only if (G', p') is isostatic on $\mathcal{M}$.*

Proof. The rigidity matrix $R_{\mathcal{M}}(G', p')$ contains 3 new rows and 3 new columns and those columns are zero everywhere in the $|E| + |V|$ rows for G. Reorder the rows and columns so that the first three rows and columns are the new ones. By the generic location of p_v and the block upper triangular structure the first three rows are independent of the rest. Thus there is a row dependency in $R_{\mathcal{M}}(G', p')$ if and only if there is a row dependency in $R_{\mathcal{M}}(G, p)$. □

The preservation of independence and isostaticity under the Henneberg 2 move is considerably more subtle. To see an aspect of this let $\mathcal{M}$ be the cylinder surface $x^2 + y^2 = 1$

in $\mathbb{R}^3$. Examining the form of $R_{\mathcal{M}}(G,p)$ it is evident that the addition of a degree 0 vertex increases the rank of the matrix by 1. Elementary linear algebra also shows that the addition of a degree 1 framework point (x_1,y_1,z_1) incident to a point (x,y,z) increases the rank by 2 if and only if (x_1,y_1,z_1) is not equal to (x,y,z) or $(-x,-y,z)$. More surprisingly, similar considerations show that there are only four possible points where the rank does not fully increase for a new degree 2 vertex. This is contrary to the situation in the plane where any point on a line through the existing edge will create a copy of K_3 whose rows give a minimal linear dependency. In view of this, adapting a typical Henneberg 2 argument for rigidity preservation would require placing the new degree 3 vertex at one of a finite number of points. However, the dependencies created by each of these points are not amenable to a standard Henneberg 2 argument.

Such difficulties motivated us in [13, Section 4], to show that continuous rigidity was preserved. The proof there makes particular use of the two trivial motions of the cylinder and it is not clear how one might generalise this method to surfaces with fewer trivial motions. Our first new approach below is based on the convergence of specialised frameworks. A direct algebraic proof is given in Section 6.

The following notation and observations will be useful.

Let $\mathcal{T}_{q_i}$ denote the tangent space for the point q_i on the surface $\mathcal{M}$ and for a framework vector $q=(q_1,q_2,\dots,q_n)$ let

$$\mathcal{T}_q=\mathcal{T}_{q_1}\oplus\mathcal{T}_{q_2}\oplus\cdots\oplus\mathcal{T}_{q_n}\subseteq\mathbb{R}^3\oplus\cdots\oplus\mathbb{R}^3=\mathbb{R}^{3n}$$

denote the joint tangent space. This is the space of infinitesimal velocities of any framework with framework vector q. With the usual Euclidean structure we have orthogonal projections

$$P_q:\mathbb{R}^{3n}\to\mathcal{T}_q,\ F_q:\mathbb{R}^{3n}\to\mathcal{F}(G,q),\ Q_q:\mathbb{R}^{3n}\to\mathcal{R}_q,$$

where $\mathcal{F}(G,q)$ is the vector space of infinitesimal flexes of the framework (G,q) and where $\mathcal{R}_q$ is the subspace of $\mathcal{T}_q$ of rigid motion infinitesimal flexes. We have $\mathcal{R}_q=\ker R_{\mathcal{M}}(K_n,q)$ for $n\geq 3$. Since $\mathcal{M}$ is smooth the function $q\to P_q$ is continuous in a neighbourhood of p. This also holds in the case of a *degenerate framework vector* p, in the sense that some or all of the vectors p_i may agree. Similarly, if the spaces $\mathcal{R}_q$ have dimension k throughout a neighbourhood of a (possibly degenerate) framework vector p then the function $q\to Q_q$ is continuous, as long as the degeneracy includes three non-colinear points. In general the function $q\to F(G,q)$ is lower semi-continuous.

Lemma 4.2. *Let G be a simple graph, let G' be derived from G by a Henneberg 2 move and let $\mathcal{M}$ be an irreducible surface. If G is minimally infinitesimally rigid on $\mathcal{M}$ then G' is minimally infinitesimally rigid on $\mathcal{M}$.*

Proof. Let (G,p) be generic on $\mathcal{M}$ with $p=(p_1,\dots,p_n)$ and let $p'=(p_0,p)$ where (G',p') is generic on $\mathcal{M}$. We let v_1v_2 denote the edge involved in the Henneberg move and write v_0 for the new vertex. Suppose that (G',p') is not infinitesimally rigid on $\mathcal{M}$. Then it follows that every specialised framework on $\mathcal{M}$ with graph G' is infinitesimally flexible. Figure 5 indicates a sequence of specialisations (G',p^k) in which only the p_0 framework point is specialised to a point p_0^k. Also p_0^k tends to p_2 in the direction a where a is a tangent vector at p_2 which is orthogonal to a tangent vector b at p_2 where b is orthogonal to p_2-p_1. More precisely, the normalised vector $(p_2^k-p_0)/\|p_2^k-p_0\|$ converges to a, as $k\to\infty$. Each

of the frameworks (G', p^k) has a unit norm flex u^k which is orthogonal to the space of rigid motion infinitesimal flexes of its framework, (G', p^k). By the Bolzano-Weierstrass theorem there is a subsequence of the sequence u^k which converges to a vector, u^∞ say, of unit norm. Discarding framework points and relabelling we may assume this holds for the original sequence. The limit flex of the degenerate framework (G', p^∞) is denoted

$$u^\infty = (u_0^\infty, u_1, u_2, \ldots, u_n).$$

Also we have

$$p^\infty = (p_2, p_1, p_2, p_3, \ldots, p_n).$$

We claim that the velocities u_1, u_2 give an infinitesimal flex of p_1p_2 (as a single edge framework on $\mathcal{M}$). To see this note that in view of the well-behaved convergence of p_0^k to p_2 (in the a direction) it follows that the velocities u_2 and u_0^∞ have the same component in the a direction, and so $(u_2 - u_0^\infty).a = 0$. Since $u_2 - u_0^\infty$ is tangential to $\mathcal{M}$ it follows from the choice of a that $u_2 - u_0^\infty$ is orthogonal to $p_2 - p_1$. On the other hand $u_1 - u_0^\infty$ is orthogonal to $p_2 - p_1$ and so taking differences $u_2 - u_1$ is orthogonal to $p_2 - p_1$ as desired.

It follows now that the restriction

$$u^\infty_{\text{res}} = (u_1, u_2, \ldots, u_n)$$

is a unit norm infinitesimal flex of (G, p) and orthogonal to the rigid motion flexes of (G, p). Thus G is dependent, as required. □

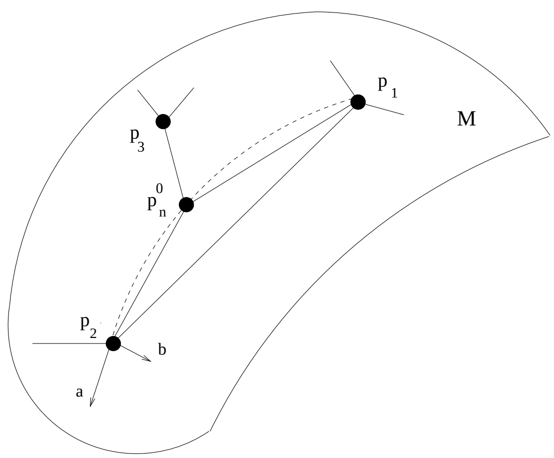

FIGURE 5. Geometry for the sequential specialisation proof of rigidity preservation under the Henneberg 2 move.

5. VERTEX SURGERY MOVES ON FRAMEWORKS

For irreducible surfaces $\mathcal{M}$, of type $2, 1$ and 0, we show the preservation of the independence of G on $\mathcal{M}$ under the vertex splitting move and the vertex-to-K_4 move and the vertex-to-4-cycle move. The form of argument in the proofs is in terms of flexibility preservation for the inverse move and this is obtained once again by the consideration of certain well-behaved contraction sequences $(G', p^k) \to (G', p^\infty)$.

5.1. **Vertex Splitting.** Recall that in a vertex-splitting move $G \to G'$ a vertex v is doubled to v_1 and v_2 with a doubling of an edge vu to v_1u, v_2u, with the remaining edges to v distributed arbitrarily between v_1 and v_2. The reverse move $G' \to G$ is an admissible triangle contraction in the sense that the result of collapsing an edge to coincident endpoints results in a simple graph.

Lemma 5.1. *Let $\mathcal{M}$ be an irreducible surface of type $2, 1$ or 0 and let $G \to G'$ be a vertex splitting move. If G is independent on $\mathcal{M}$ then G' is independent on $\mathcal{M}$.*

Proof. To fix notation, let G have $n-1$ vertices $v, v_3, v_4, \dots, v_n$ with vertex v to be split and v_3 featuring as the vertex u in the vertex-splitting move. Let p, p' be generic vectors for G, G' respectively, with $p = (p(v), p(v_3), \dots, p(v_n)) = (p_1, p_3, \dots, p_n)$ and $p' = (p(v_1)', p(v_2)', \dots, p(v_n)') = (p_1, p_2, \dots, p_n)$.

It will be sufficient to show that if G' is dependent on $\mathcal{M}$ then so too is G. Accordingly we assume that (G', p') is a generic framework which is infinitesimally flexible. Let p_2^k be a sequence of generic points on $\mathcal{M}$ which converges to p_1 in the following well-behaved manner, namely the unit vectors a_k in the directions $p_2^k - p_1$ converge to a unit vector a with $a.(p_3 - p_1) \neq 0$. Also, let $p^k = (p_1, p_2^k, p_3, \dots, p_n)$ and let $p^\infty = (p_1, p_1, p_3, \dots, p_n)$. By the assumption, for each k there exists a unit vector $u^k = (u_1^k, \dots, u_n^k)$ in $\mathcal{T}_{p^k}$ which is an infinitesimal flex of (G', p^k) and which has no rigid motion flex component, in the sense that $Q_{p^k} u^k = 0$. Furthermore, taking subsequences, we may assume that u^k converges to some unit norm flex $u' = (u_1, \dots, u_n)$ as k tends to infinity. By the flex condition we have

$$u_1.(p_1 - p_3) = u_3.(p_1 - p_3)$$

and

$$u_2.(p_1 - p_3) = \lim_{k\to\infty} u_2^k.(p_2^k - p_3) = \lim_{k\to\infty} u_3^k.(p_2^k - p_3) = u_3.(p_1 - p_3).$$

On the other hand, since a_k is a scalar multiple of $p_2^k - p_1$,

$$(u_2 - u_1).a = \lim_{k\to\infty} (u_2^k - u_1^k).a_k = \lim_{k\to\infty} (u_2^k - u_1^k).(p_2^k - p_1) = 0.$$

Thus $u_1 = u_2$ since both vectors lie in the same tangent plane and their difference is orthogonal to the linearly independent vectors a and $p_1 - p_2$.

Note that there is always a natural injective map

$$\iota : \mathcal{F}(G, p) \to \mathcal{F}(G', p^\infty),$$

provided by the map $(w_1, w_3, w_4, \dots, w_n) \to (w_1, w_1, w_3, w_4, \dots, w_n)$. Also a flex of (G', p^∞) of the form $(w_1, w_1, w_3, w_4, \dots, w_n)$ determines the flex $(w_1, w_3, w_4, \dots, w_n)$ of (G, p). Thus the flex u' gives rise to a flex, u say, of (G, p). From the continuity of $q \to Q_q$ it follows that $Q_{p^\infty} u' = 0$. Thus $Q_p u = 0$ and so u is a proper flex and G is dependent, as required. $\square$

The preservation of independence under vertex splitting for bar-joint frameworks in various dimensions was shown by Whiteley [22]. The arguments there were based on self-stresses and k-frames. Here we argue somewhat more directly in terms of infinitesimal flexes.

5.2. **Vertex-to-K_4 move.** We now show preservation of independence under the vertex-to-K_4 move $G \to G'$. Once again we will argue in terms of a *well-behaved contraction* $(G', p^k) \to (G', p^\infty)$, this time with respect to a particular sequence of subframeworks $(K_4, (p_1, p_2^k, p_3^k, p_4^k))$ with $p_i^k \to p_1$ as $k \to \infty$, for each $i = 2, 3, 4$.

First we consider separately the case of a cylinder since here we may exploit the two trivial infinitesimal motions to obtain a simple direct proof.

Lemma 5.2. *Let $\mathcal{M}$ be the cylinder, let G' be $(2,2)$-sparse with proper subgraph K_4 and let $p = (p_1, \ldots, p_n)$ be a framework vector in $\mathcal{M}^n$ for G', with $p_1, \ldots, p_4$ the placement of the vertices for K_4. Let*

$$p^k = (p_1, p_2^k, p_3^k, p_4^k, p_5, \ldots, p_n)$$

be a sequence in $\mathcal{M}^n$ with $p_i^k \to p_1$ as $k \to \infty$, for $i = 2, 3, 4$, and suppose that

(i) each framework $(K_4, (p_1, p_2^k, p_3^k, p_4^k))$ has 2 independent infinitesimal flexes on $\mathcal{M}$,

(ii) the dimension of the infinitesimal flex space of (G', p^k) is greater than 2 for each k.

Then there is a unit norm infinitesimal flex u' of (G', p^∞) of the form

$$u' = (0, 0, 0, 0, u_5, \ldots, u_n).$$

Proof. In view of (ii) for each k there exists an infinitesimal flex $u^k = (u_1^k, \ldots, u_n^k)$ of (G, p^k) in $\mathbb{R}^{3n}$ with norm $\|u^k\| = 1$, and with $u_1^k = 0$ for all k. Thus, from (i) it follows that $u_i^k = 0$ for $i = 2, 3, 4$. Taking subsequences we may assume that u^k converges to a unit norm flex u' of (G', p^∞) and this has the desired form. □

Corollary 5.3. *The vertex-to K_4 move on the cylinder preserves generic infinitesimal rigidity.*

Proof. Let $G \to G'$ be a vertex-to-K_4 move. Suppose that G' is dependent. Let p^k be a sequence of generic framework vectors for G' as specified in Lemma 5.2, with $p_i^k \to p_1$ for $i = 1, \ldots, 4$. Since p^k is generic (i) holds, and (ii) holds since G' is dependent. Thus, by the lemma (G', p^∞) has an infinitesimal flex of the indicated form. This in turn gives an infinitesimal flex u of (G, p) with $u_1 = 0$ and so G is dependent, as required. □

We now consider the much more subtle case of irreducible surfaces which are of type 1 or 0. In this case we exploit the characteristic property that at a generic point p_1 on $\mathcal{M}$ the principal curvatures are distinct.

Let $(K_4, (p_1, \ldots, p_4))$ be a bar-joint framework on the irreducible surface $\mathcal{M}$ and assume that the principal curvatures κ_s, κ_t at p_1 are well-defined, with $\kappa_s \neq \kappa_t$, and that $\hat{s}$ and $\hat{t}$ are associated orthonormal vectors in the tangent plane at p_1. For definiteness let $\hat{n}$ be the unit normal at p_1 with $\hat{s}, \hat{t}, \hat{n}$ a right-handed orthonormal triple. By Taylor's theorem, in a neighbourhood of p_1 the points p on $\mathcal{M}$ take the form

$$p(s,t) = p_1 + (s\hat{s} + t\hat{t}) + 1/2(\kappa_s s^2 + \kappa_t t^2)\hat{n} + r(s,t) \tag{5.1}$$

with $\|r(s,t)\| = O(\|(s,t)\|^3)$, for $\|(s,t)\| \leq R$, say. In particular

$$\frac{dp}{ds} = \hat{s} + \kappa_s s\hat{n} + r_s, \quad \frac{dp}{dt} = \hat{t} + \kappa_t t\hat{n} + r_t$$

where $\|r_s\|$ and $\|r_t\|$ are of order $\|(s,t)\|^2$. Also the vectors

$$n(s,t) = dp/ds(s,t) \times dp/dt(s,t) \tag{5.2}$$

give a continuous choice of normal vectors in a neighbourhood of p_1, with

$$n(s,t) = \hat{s}\times\hat{t} + \kappa_t t\hat{s}\times\hat{n} + \kappa_s s\hat{n}\times\hat{t} + \underline{r}$$
$$= \hat{n} - (\kappa_t t\hat{t} + \kappa_s s\hat{s}) + \underline{r}$$

where $\|\underline{r}\|$ is of order $\|(s,t)\|^2$ in this neighbourhood.

Assume now that the triple p_2, p_3, p_4 is generic and lies in this neighbourhood with $p_i = p(s_i, t_i)$. For $\epsilon \leq 1$ let $p_i^\epsilon = p(\epsilon s_i, \epsilon t_i)$. By a *well-behaved K_4 contraction* of (G,p) over the subgraph $K_4 \subseteq G$, with vertices $v_1, \dots, v_4$, we mean a framework sequence (G, p^k) with

$$p^k = (p_1, p_2^{\epsilon_k}, p_3^{\epsilon_k}, p_4^{\epsilon_k}, p_5, \dots, p_n),$$

where $\epsilon_k \to 0$ as $k \to \infty$ and where the the local coordinates $s_2, t_2, s_3, t_3, s_4, t_4$ satisfy the determinant condition

$$\begin{vmatrix} s_2 & t_2 & s_2t_2 \\ s_3 & t_3 & s_3t_3 \\ s_4 & t_4 & s_4t_4 \end{vmatrix} \neq 0.$$

It is straightforward to see that we can choose a well-behaved K_4 contraction. For example if $s_i = i, t_i = i^2$ for $i = 2,3,4$ then the determinant has the value 48.

Lemma 5.4. *Let $(K_4, (p_1, p_2^k, p_3^k, p_4^k)), k = 1, 2, \dots,$ be a well-behaved contraction of frameworks on $\mathcal{M}$ and let $u_k, k = 1, 2, \dots,$ be an associated sequence of infinitesimal flexes which forms a convergent sequence in $\mathbb{R}^{12}$. Then the limit vector has the form (u_1, u_1, u_1, u_1).*

Proof. Let $u = (u_1, \dots, u_4)$ be an infinitesimal flex of (K_4, p). Equivalently, $u_i.n_i = 0$ where n_i is the unit normal at p_i and $(p_i - p_j).(u_i - u_j) = 0$ for $1 \leq i < j \leq 4$. Since (K_4, p) is infinitesimally rigid in $\mathbb{R}^3$ the flex u is equal to $u_a + u_b$ where u_b is determined by translation by the vector b and where u_a corresponds to an infinitesimal rotation about a line through p_1 with direction vector a. Thus $u_1 = b$ and we may choose the magnitude and direction of a so that $u_i - u_1 = (p_i - p_1) \times a$, for $i = 2,3,4$. Substituting gives

$$(a \times (p_i - p_1)).n_i = u_1.n_i,$$

or equivalently,

$$a.(n_i \times (p_i - p_1)) + b.n_i = 0,$$

for $i = 2,3,4$.

We have the normal vectors $n(s,t) = dp/ds(s_i,t_i) \times dp/dt(s_i,t_i)$ as in Equation 5.2 above. At the point $p_i^\epsilon = p(\epsilon s_i, \epsilon t_i)$ these normals take the form

$$n_i^\epsilon = n(\epsilon s_i, \epsilon t_i) = \hat{n} - \epsilon(\kappa_t t_i \hat{t} + \kappa_s s_i \hat{s}) + \underline{r}_i^\epsilon$$

where $\|\underline{r}_i^\epsilon\| = O(\epsilon^2)$.

Consider now an infinitesimal flex u^ϵ of the framework (K_4, p^ϵ) on $\mathcal{M}$. The associated equations are

$$a^\epsilon.(n_i^\epsilon \times (p_i^\epsilon - p_1)) + b^\epsilon.n_i = 0,$$

for $i = 2,3,4$, and we may identify the crossed product here as

$$n_i^\epsilon \times (p_i^\epsilon - p_1) = (\hat{n} - \epsilon(\kappa_t t_i \hat{t} + \kappa_s s_i \hat{s})) \times (\epsilon(s_i\hat{s} + t_i\hat{t}) + 1/2\epsilon^2(\kappa_s s_i^2 + \kappa_t t_i^2)\hat{n}) + R_i^\epsilon$$

with $\|R_i^\epsilon\| = O(\epsilon^3)$.

We may assume, by passing to a subsequence, that ϵ runs through a sequence ϵ_k tending to zero and that associated unit norm flexes u^ϵ converge to a limit flex u^0 of the degenerate

framework $(K_4,(p_1,p_1,p_1,p_1))$ on $\mathcal{M}$. Let $b^0=u_1^0$ and let b^ϵ and a^ϵ be the associated vectors. While $b^\epsilon=u_1^\epsilon$ converges to b^0, as $\epsilon=\epsilon_k\to 0$, the sequence (a^{ϵ_k}) may be unbounded. However, in view of the three equations

$$u_i^\epsilon-u_1^\epsilon=(p_i^\epsilon-p_1)\times a^\epsilon$$

and the definition of p_i^ϵ it follows that $\|a^{\epsilon_k}\|$ is at worst of order $1/\epsilon_k$. We shall show that $\|a^{\epsilon_k}\|$ is in fact bounded and so, from the equation above, the desired conclusion follows.

Returning to the three equations which determine a^ϵ from b^ϵ we have

$$a^\epsilon.(s_i\hat{t}-t_i\hat{s}-\epsilon s_it_i(\kappa_s-\kappa_t)\hat{n}+R_i^\epsilon)-\kappa_s(b^\epsilon.\hat{s})s_i-\kappa_t(b^\epsilon.\hat{t})t_i+r_i^\epsilon=0.$$

Note that $a^\epsilon.R_i^\epsilon=O(\epsilon^2)$ and so it follows, introducing coordinates for a^ϵ, that

$$(a_s^\epsilon\hat{s}+a_t^\epsilon\hat{t}+a_n^\epsilon\hat{n}).(s_i\hat{t}-t_i\hat{s}-\epsilon s_it_i(\kappa_s-\kappa_t)\hat{n})-\kappa_s(b^\epsilon.\hat{s})s_i-\kappa_t(b^\epsilon.\hat{t})t_i=O(\epsilon^2),$$

for $i=2,3,4$. Thus

$$-a_s^\epsilon t_i+a_t^\epsilon s_i-a_n^\epsilon\epsilon s_it_i(\kappa_s-\kappa_t)=d_i^\epsilon,\quad \text{for } i=2,3,4,$$

where

$$d_i^\epsilon=b^\epsilon.(\kappa_ss_i\hat{s}+\kappa_tt_i\hat{t})+X_i^\epsilon,$$

with $X_i^\epsilon=O(\epsilon^2)$.

Let $\eta_i=\epsilon s_it_i(\kappa_s-\kappa_t)$ for $i=2,3,4$, let A_ϵ be the matrix

$$\begin{bmatrix} -t_2 & s_2 & -s_2t_2\eta_2\\ -t_3 & s_3 & -s_3t_3\eta_3\\ -t_4 & s_4 & -s_4t_4\eta_4\end{bmatrix},$$

and note that $\det A_\epsilon=C\epsilon$ for some nonzero constant C. By Cramer's rule we have

$$a_n^\epsilon=(\det A_\epsilon)^{-1}\begin{vmatrix} -t_2 & s_2 & d_2^\epsilon\\ -t_3 & s_3 & d_3^\epsilon\\ -t_4 & s_4 & d_4^\epsilon\end{vmatrix}=(\det A_\epsilon)^{-1}\begin{vmatrix} -t_2 & s_2 & X_2^\epsilon\\ -t_3 & s_3 & X_3^\epsilon\\ -t_4 & s_4 & X_4^\epsilon\end{vmatrix},$$

since the column for $d_i^\epsilon-X_i^\epsilon$ is a linear combination of the first two columns. It follows that the sequence $a_n^{\epsilon_k}$ is bounded.

The boundedness of $(a_s^{\epsilon_k})$, and similarly $(a_t^{\epsilon_k})$, follows more readily, since

$$a_s^\epsilon=(\det A_\epsilon)^{-1}\begin{vmatrix} d_2^\epsilon & s_2 & -s_2t_2\eta_2\\ d_3^\epsilon & s_3 & -s_3t_3\eta_3\\ d_4^\epsilon & s_4 & -s_4t_4\eta_4\end{vmatrix}$$

and each of the η_i have a factor ϵ. Thus, the sequence of vectors a^{ϵ_k} is bounded, as desired. □

Lemma 5.5. *Let $\mathcal{M}$ be an irreducible surface of type 1, let G' be $(2,1)$-sparse with $v_1,\dots,v_4$ inducing a K_4 subgraph, and let $p=(p_1,\dots,p_n)$ be a generic framework vector in $\mathcal{M}^n$. Let*

$$p^k=(p_1,p_2^k,p_3^k,p_4^k,p_5,\dots,p_n)$$

be sequence in $\mathcal{M}^n$, with $p_i^k\to p_1$ as $k\to\infty$, for $i=2,3,4$, such that (G,p^k) is a well-behaved contraction with limit (G',p^∞). If the rigid infinitesimal motion spaces $\mathcal{R}_{p^k}$ and $\mathcal{R}_{p^\infty}$ are one-dimensional and the dimension of the flex space $\mathcal{F}(G',p^k)$ is greater than 1 for all k, then there is a unit norm flex u in $\mathcal{F}(G',p^\infty)$ which is orthogonal to $\mathcal{R}_{p^\infty}$ and satisfies $u_1=u_2=u_3=u_4$.

Proof. By the hypotheses for each k there exists an infinitesimal flex $u^k = (u_1^k, \dots, u_n^k)$ of (G, p^k) lying in the multiple tangent space $\mathcal{T}_{p^k}$ such that the Euclidean norm of u^k is unity and u^k is orthogonal to the subspace $\mathcal{R}_{p^k}$. Taking a subsequence if necessary we may assume that u^k converges to u as $k \to \infty$. By Lemma 5.4 the velocities $u_1, \dots, u_4$ agree. By the hypotheses, the orthogonal projections Q_k onto $\mathcal{R}_{p^k}$ converge to the projection Q_∞ onto $\mathcal{R}_{p^\infty}$ and so u is orthogonal to $\mathcal{R}_{p^\infty}$, as desired. □

Corollary 5.6. *The vertex-to K_4 move for a type* 1 *surface preserves generic infinitesimal rigidity.*

Proof. This follows from the previous lemma in the same manner as the proof of Corollary 5.3. □

5.3. **The vertex-to-4-cycle move.**

Lemma 5.7. *Let $\mathcal{M}$ be an irreducible surface of type k and let $G \to G'$ be a vertex-to-*4*-cycle move. If G is minimally infinitesimally rigid on $\mathcal{M}$ then G' is minimally infinitesimally rigid on $\mathcal{M}$.*

Proof. Once again we use a sequential contraction argument. Let G have n vertices $v_1, v_2, , \dots, v_n$ and edges v_1v_2, v_1v_3 and let $G \to G'$ be the move in question, with new vertex v_0 and edges v_0v_2, v_0v_3. It will be sufficient to show that if G' is dependent on $\mathcal{M}$ then so too is G.

Let p, p' be the generic framework vectors for G, G' respectively, with $p' = (p_0, p_1, \dots, p_n)$. Also let $p^k = (p_0^k, p_1, \dots, p_n)$ be generic, with p_0^k converging p_1. By the assumption for each k there exists a unit vector $u^k = (u_0^k, u_1^k, \dots, u_n^k)$ in the joint tangent space $\mathcal{T}_{p^k}$ which is an infinitesimal flex of (G', p^k) and which is orthogonal to the rigid motion flexes. In earlier notation, $Q_{p^k}u^k = 0$. Taking subsequences, we may assume that u^k converges to some unit norm flex $u' = (u_0, u_1, \dots, u_n)$ of the degenerate framework (G', p^∞), as k tends to infinity, where $p^\infty = (p_1, p_1, p_2, \dots, p_n)$. Also, by the assumption on G, this degenerate framework (for G') has a space of rigid motion flexes which is naturally identifiable with the space of rigid motion flexes of (G, p). It remains to show that $u_0 = u_1$ so that we may conclude that $(u_1, \dots, u_n)$ is a proper flex of (G, p), completing the proof.

It follows from the flex conditions and taking limits that $u_0 - u_2$ is orthogonal to $p_1 - p_2$, and $u_0 - u_3$ is orthogonal to $p_1 - p_3$. Also $u_1 - u_2$ is orthogonal to $p_1 - p_2$, and $u_1 - u_3$ is orthogonal to $p_1 - p_3$. It follows, subtracting, that $u_0 - u_1$ is orthogonal to $p_1 - p_2$ and to $p_2 - p_3$. At the same time $u_0 - u_1$ lies in the tangent plane at p_1 and we may choose p_2, p_3 so that 0 is the only tangent vector orthogonal to $p_1 - p_2$ and to $p_2 - p_3$.

□

6. The algebraic approach

We now give a direct algebraic proof of the preservation of infinitesimal rigidity under the Henneberg 2 move on an irreducible surface. We expect this approach to be more widely useful in the analysis of bar-joint frameworks in higher dimensions.

Assume that $\mathcal{M}$ is an irreducible surface of type k which is defined by the irreducible polynomial $m(x, y, z) = 0$ where the coefficients of m are in $\mathbb{Q}$. Suppose that G is a $(2, k)$-tight graph and that (G, p) is a generic framework on $\mathcal{M}$ with $p = (p_1, \dots, p_n)$. Also

let $p^+ = (p, p_v)$ where (G^+, p^+) is generic on $\mathcal{M}$. We write v_1v_2 for the edge involved in the Henneberg move and v for the new vertex.

Since G is independent the rigidity matrix $R_{\mathcal{M}}(G\backslash v_1v_2, p)$ has a flex vector $u = (u_1, \dots, u_n)$ in the nullspace which is not a flex of (G, p). In particular $(p_1 - p_2).(u_1 - u_2) \neq 0$. Moreover we may choose u as a solution of the equations $R_{\mathcal{M}}(G, p)u = A$ where A is a column vector with all entries zero except for an entry of unity in the row representing the edge v_1v_2. This gives a set of linear equations with coefficients in $\mathbb{Q}(p)$ and we can select a solution for which all coordinates of the velocities u_i lie in $\mathbb{Q}(p)$.

We show first that u does not extend to a flex of (G, p^+).

Suppose by way of contradiction that $u^+ = (u, u_v)$ is an extension of u to a flex of (G^+, p^+) with component u_v acting at p_v. Introducing the notation $p_{i,j} = p_i - p_j$, $p_{v,i} = p_v - p_i$ and similarly $u_{i,j} = u_i - u_j$, $u_{v,i} = u_v - u_i$ the flex u_v satisfies four equations

$$p_{v,i}.u_{v,i} = 0, \quad 1 \leq i \leq 3, \quad u_v.N(p_v) = 0,$$

where $N(p_v)$ is the normal to the surface $\mathcal{M}$ at p_v given by

$$N(p_v) = (\nabla m)(p_v) = (\partial m/\partial x, \partial m/\partial y, \partial m/\partial z)|_{p_v}.$$

Introducing the coordinate notation $(p^x_{v,1}, p^y_{v,1}, p^z_{v,1})$ for $p_{v,1}$ these four equations for the three components of $u_{v,1}$ have a consistent solution if and only if $\det(D) = 0$, where

$$D = \begin{bmatrix} p^x_{v,1} & p^y_{v,1} & p^z_{v,1} & 0 \\ p^x_{2,1} & p^y_{2,1} & p^z_{2,1} & -u_{2,1}.p_{v,2} \\ p^x_{3,1} & p^y_{3,1} & p^z_{3,1} & -u_{3,1}.p_{v,3} \\ N(p_v)^x & N(p_v)^y & N(p_v)^z & u_1.N(p_v) \end{bmatrix}.$$

Let $P_v = (x, y, z)$ be the vector of indeterminates corresponding to p_v, let $P_{v,i} = P_v - p_i$, $i = 1, 2, 3$, and let

$$D(P_v) = D(x, y, z) = \begin{bmatrix} P^x_{v,1} & P^y_{v,1} & P^z_{v,1} & 0 \\ P^x_{2,1} & P^y_{2,1} & P^z_{2,1} & -u_{2,1}.p_{v,2} \\ P^x_{3,1} & P^y_{3,1} & P^z_{3,1} & -u_{3,1}.p_{v,3} \\ N(P_v)^x & N(P_v)^y & N(P_v)^z & u_1.N(P_v) \end{bmatrix}.$$

Then the polynomial $\det(D(P_v))$ lies in the ring $\mathbb{Q}(p)[P_v]$. Since

$$0 = \det(D) = \det(D(P_v))|_{p_v}$$

the polynomial $\det(D(P_v))$ evaluates to zero under the substitution $P_v = p_v$. Since p_v is generic on $\mathcal{M}$ this implies that $\det(D(P_v))$ is in the ideal of $\mathbb{Q}(p)[P_v]$ generated by the surface polynomial $m(x, y, z)$. Thus $\det(D(P_v)) = h(P_v)m(P_v)$ for some polynomial $h(P_v)$ in $\mathbb{Q}(p)[P_v]$.

Since $\det(D(P_v)) = h(P_v)m(P_v)$ and $\nabla(m(P_v)) = N(P_v)$ we have

$$\nabla(det(D(P_v))) = h(P_v)N(P_v) + \nabla(h(P_v))m(P_v)$$

and so

$$\nabla(\det(D(P_v)))|_{p'_v} = h(p'_v)N(p'_v)$$

for any point p'_v satisfying $m(p'_v) = 0$. This implies $a.\nabla(\det(D(P_v)))|_{p'_v} = 0$ for any $a \in \mathbb{R}^3$ satisfying $a.N(p'_v) = 0$ and any point p'_v satisfying $m(p'_v) = 0$. We consider $p'_v = p_1$ which satisfies this property.

We have $u_1.N(p_1) = 0$. Also, since the first row of the matrix $D(P_v)|_{p_1}$ is zero we get a non-zero contribution to $\nabla(\det(D(P_v)))|_{p_1}$ only from the action of the ∇ operator on the first row of $D(P_v)$. Thus, in vector form, $\nabla(\det(D(P_v)))|_{p_1}$ is the determinant of the matrix

$$\begin{bmatrix} i & j & k & 0 \\ p^x_{2,1} & p^y_{2,1} & p^z_{2,1} & -u_{2,1}.p_{1,2} \\ p^x_{3,1} & p^y_{3,1} & p^z_{3,1} & -u_{3,1}.p_{1,3} \\ N(p_1)^x & N(p_1)^y & N(p_1)^z & u_1.N(p_1) \end{bmatrix}.$$

Expanding the determinant along the final column gives

$$\nabla(\det(D(P_v)))|_{p_1} = ((p_{2,1}.u_{2,1})p_{3,1} \times N(p_1) - (p_{3,1}.u_{3,1})p_{2,1} \times N(p_1))$$

and so from the above

$$a.((p_{2,1}.u_{2,1})p_{3,1} \times N(p_1) - (p_{3,1}.u_{3,1})p_{2,1} \times N(p_1)) = 0$$

for all a with the property that $a.N(p_1) = 0$.

The vector $a = N(p_1) \times (p_{2,1} \times N(p_1))$ satisfies $a.N(p_1) = 0$ and $a.\nabla(\det(D(P_v)))|_{p_1} = 0$ gives the condition $(p_{2,1}.u_{2,1})b = 0$ where

$$b = (N(p_1) \times (p_{2,1} \times N(p_1))).(p_{3,1} \times N(p_1)) = N(p_1).(p_{3,1} \times p_{2,1}).$$

We have $b \neq 0$ because the condition that $N(p_1).(p_{3,1} \times p_{2,1}) = 0$ for all p_2, p_3 on $\mathcal{M}$ contradicts the smoothness requirement that $p_{3,1} \times p_{2,1}$ becomes parallel to $N(p_1)$ for p_2 and p_3 close to p_1. Thus $p_{2,1}.u_{2,1} = 0$ which is contrary to our original choice of u and so we conclude that u does not extend to a flex of (G^+, p^+).

On the other hand, suppose that a flex $u = (u_1, \dots, u_n)$ for of $(G\backslash v_1v_2, p)$ on $\mathcal{M}$ does extend to a flex (u, u_v) of (G^+, p^+) on $\mathcal{M}$. Then u_v is the solution of the three equations

$$u_v.(p_v - p_1) = u_1.(p_v - p_1), u_v.(p_v - p_2) = u_2.(p_v - p_2) \text{ and } u_v.N(p_v) = 0$$

and the solution is unique because $(p_v - p_1) \times (p_v - p_2).N(p_v) \neq 0$ for generic p_v, p_1, p_2 for the same reason given above that $b \neq 0$. Also if u is zero then (u, u_v) is zero and so every flex in the nullspace of $R_M(G^+, p^+)$ is the extension of a flex of $R_{\mathcal{M}}(G\backslash v_1v_2, p)$.

Finally, consider the matrix $R' = R_M(G\backslash v_1v_2, p)$ of size $m' \times n'$. Since G is independent we have $m' + 1 \leq n'$ and $\text{rank}(R') = m'$. For the matrix $R = R_{\mathcal{M}}(G^+, p^+)$ of size $m \times n$ we have $m = m' + 4$ and $n = n' + 3$ and so $m \leq n$ and $n - m = n' - m' - 1$. Every flex of R' either does not extend to a flex of R or extends to a unique flex of R and every flex of R is the extension of some flex of R'. Thus $|\operatorname{null}(R')| > |\operatorname{null}(R)|$. By Lemma 6.1 the rank of R is m which means that G^+ is independent on $\mathcal{M}$, as required.

Lemma 6.1. *Let R be an $m \times n$ matrix with $m \leq n$ and R' an $m' \times n'$ matrix with $m' \leq n'$ and* $\text{rank}(R') = m'$. *If $n - m = n' - m' - 1$ and $|\operatorname{null}(R)| < |\operatorname{null}(R')|$ then* $\text{rank}(R) = m$.

Proof. $\text{rank}(R) = n - |\operatorname{null}(R)| \geq n - |\operatorname{null}(R')| + 1$ and $|\operatorname{null}(R')| = n' - m'$ so $\text{rank}(R) \geq m$. □

7. Minimal rigidity on type 1 Surfaces

There is a final independence preserving move that we need for the proof of the main result. Recall that if G and H are graphs with vertices $g \in G, h \in H$ then the *edge joining move* combines G and H by adding the edge gh.

For the plane or the sphere, if G and H have minimally rigid generic realisations then $R_{\mathcal{M}}(G,p)$ and $R_{\mathcal{M}}(H,q)$ have 3-dimensional nullspaces. Thus $R_{\mathcal{M}}(G \cup H,(p,q))$ has a 6-dimensional nullspace and adding gh adds at most one to the rank. Similarly, infinitesimal rigidity is not preserved in the case of the cylinder since the nullspace is at least 3-dimensional. However, we have the following.

Lemma 7.1. *Let $\mathcal{M}$ be an irreducible surface of type 1. Let (G,p) and (H,q) be minimally infinitesimally rigid on $\mathcal{M}$, and let G' be an edge join of G and H. If $|V(G)|$ and $|V(H)|$ are greater than 4 then $(G',(p,q))$ is minimally infinitesimally rigid on $\mathcal{M}$.*

Proof. Consider the block matrix form

$$R_{\mathcal{M}}(G',(p,q)) = \begin{bmatrix} R_{\mathcal{M}}(G,p) & 0 \\ * & * \\ 0 & R_{\mathcal{M}}(H,q) \end{bmatrix}.$$

By the hypotheses the nullspaces of the rigidity matrices $R_{\mathcal{M}}(G,p)$ and $R_{\mathcal{M}}(H,q)$ are one-dimensional. Let $u=(u_p,u_q)$ be an infinitesimal flex of the edge-joined framework and let gh be the joining edge for G'. Subtracting a tangential rigid motion infinitesimal flex we may assume that u_p assigns a zero velocity to the framework joint p_g. Since u_p is a flex of (G,p) on $\mathcal{M}$ it follows that $u_p=0$. Also in view of the row in $R_{\mathcal{M}}(G',(p,q))$ for the joining edge it follows that u_q assigns zero velocity to the framework vertex q_h. Thus, since u_q is in the nullspace of $R_{\mathcal{M}}(H,q)$ it follows that $u_q=0$. Thus $u=0$ and the nullspace of $R_{\mathcal{M}}(G',(p,q))$ has dimension one, as desired. □

We now arrive at the proof of our main result, Theorem 1.1. For the readers convenience we first re-state the theorem.

Theorem 7.2. *Let $G=(V,E)$ be a simple graph and let $\mathcal{M}$ be an irreducible surface of type 1. Then a generic framework (G,p) on $\mathcal{M}$ is isostatic if and only G is K_1,K_2,K_3,K_4 or is $(2,1)$-tight.*

Proof. That the underlying graph of an isostatic framework on $\mathcal{M}$ is $(2,1)$-tight or is a small complete graph follows from Theorem 2.6. For the sufficiency direction one can check that the minimal graph $K_5\backslash e$ in the inductive characterisation of $(2,1)$-tight graphs is isostatic on $\mathcal{M}$. The sufficiency of $(2,1)$-tightness now follows from Theorem 1.2 if minimal generic rigidity is preserved by Henneberg 1 and 2 moves, the vertex-to-K_4 move, the vertex-to-4-cycle move and the edge joining move. This is the content of Lemma 4.1, Lemma 4.2, Corollary 5.6, Lemma 5.7 and Lemma 7.1. □

Note that we could also have used Theorem 2.13, applying Lemma 5.1, to prove the theorem.

We finish by noting some further natural considerations for frameworks constrained to surfaces.

The usual two-dimensional torus embedded in 3-dimensional space has freedom type 1 and so isostatic frameworks on this surface are characterised as in the previous theorem. When the torus is realised in $\mathbb{R}^4$ one may also consider the *Clifford torus* $\mathcal{T}$, that is, the real algebraic variety and smooth manifold defined by the polynomial equations $x^2+y^2=1$ and $z^2+w^2=1$. The definition of type (freedom number) given in Definition 2.2 extends without change to an algebraic surface $\mathcal{M}$ in $\mathbb{R}^d$.

Definition 7.3. *An embedded manifold* $\mathcal{M}$ *in* $\mathbb{R}^d$ *is of type* k *if* $\dim\ker R_{\mathcal{M}}(K_n, q) \geq k$ *for all framework* (K_n, p) *on* $\mathcal{M}$*, for* $n = 2, 3, \dots,$ *and* k *is the largest such integer.*

In particular the Clifford torus has freedom type 2. The rigidity analysis in this setting requires us to adapt the definition of the rigidity matrix. The details are similar to those in Definition 2.1 with the following changes. There are now 4 columns per vertex and 2 rows per vertex where the rows for vertex i (and corresponding framework point (x_i, y_i, z_i, w_i)) are zero except in the 4-tuple corresponding to i where the entries in the first row are $x_i, y_i, 0, 0$ and the second are $0, 0, z_i, w_i$.

On the other hand take the product of a circle and an ellipse. This is the algebraic variety $\mathcal{S}$ defined by, say, $x^2 + y^2 = 1$ and $z^2 + w^2/2 = 1$ in $\mathbb{R}^4$. $\mathcal{S}$ admits exactly 1 trivial motion. Adapting the methods of the last section would lead to the $(2, 2)$-tight and the $(2, 1)$-tight characterisations of frameworks on $\mathcal{T}$ and $\mathcal{S}$ respectively.

Finally, it is natural to seek a similar characterisation of our main result to frameworks with vertices constrained to an irreducible surface of freedom type 0. There are a variety of such surfaces that a characterisation could apply to including an elliptical cone, a mobius strip, a hyperboloid and a hyperbolic paraboloid. Note that the graphs of rigid frameworks need not be connected in this setting. We expect that the relevant graphs will be $(2, 0)$-tight simple graphs and that the rigidity preservation methods above will be useful in deriving this characterisation. However, complications in the determination of an inductive construction for such graphs are evident at the outset, since a $(2, 0)$-tight simple graph can be 4-regular.

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